GTSG: Characteristic Classes and the Signature Theorem

Riley Moriss

May 23, 2025

1 Characteristic Classes

We take the axioms in [MS16] and [Hir66]. Given a —— rank n vector bunlde $(\xi, \pi : E \to B)$ then the ——- classes are elements

$$c_i(\xi) \in H^i(B; G)$$

The total —– class is again the formal sum

$$c(\xi) = \sum_i c_i(\xi) \in H^{\Pi}(B)$$

the ——- classes are the unique classes satisfying

1. $c_0(\xi) = 1$
2. (Naturality) Given a bundle map $f : E(\xi) \to E(\xi')$ the classes are $c_i(\xi) = \bar{f}^* c_i(\xi')$. The bar is the induced map on the base space.
3. (Respects sums) $c(\xi \oplus \xi') = c(\xi) \smile c(\xi')$
4. (Normalisation) The first —— class of the canonical bundle over $\mathbb{C}P^n$ is non-trivial (it generates).

For example for Stieffel-Whitney classes we set $G = \mathbb{Z}/2\mathbb{Z}$, real bundles and normalise via the mobius bundle over the circle. For Chern classes we take $\mathbb{Z}$ coefficients for complex bundles and normalize by a certain bundle over projective space. Chern classes will obviously be zero for complex bundles in odd degrees, so we reindex them such that $c_i \in H^{2i}$. To a real bundle we can associate the complex bundle given by tensoring the fibers with $\mathbb{C}$. This gives us the definition of the Pontryagin classes

$$p_i(\xi) = \big(-1\big)^i c_{2i}(\xi \otimes \mathbb{C})$$

We may also use $\tilde{p}$ which are just the Chern classes of the complexification (no sign adjustment). ⚠️ Note that Pontryagin classes do not satisfy additivity *on the nose* they are only additive mod 2, precisely $2\big(p(\xi \oplus \xi') - p(\xi)p(\xi')\big) = 0$. As you can see below they are still natural transformations we just dont consider the functors to be between semi-groups.

If the bundle is over a compact and oriented smooth manifold without boundary then we have fundamental class (unique up to a sign), an element of the top degree homology. Given a partition $i_1, ..., i_r$ of n the (real) dimension of the manifold, and therefore the rank of the tangent bundle we can get an integer

$$c_{i_1}(\tau_M) \cdots c_{i_r}(\tau_M)[M]$$

by evaluating the cup product of the —— classes on the fundamental class. The partition ensures that the product is in the correct degree of cohomology, any such product that lands in the right degree is valid. These are called the —— numbers, they clearly depend on the given partition.

Remark: The difference between H^{Π} and H^* as rings is that H^{Π} has no finiteness condition, that is H^* is the *direct sum* of the cohomology groups and hence any element in the direct sum will have only finitely many non-zero components. This is not the case for H^{Π}. There is no difference for finite CW complexes.

Remark: A characteristic class in general can be formulated or axiomatise as a natural transformation in the following way. First we have two functors, one is the cohomology theory and one is the relevant theory of bundles

$$\mathrm{Vect}_{-} : \mathrm{Top}^{\mathrm{op}} \to \mathrm{Set}$$

$$H^* : \mathrm{Top}^{\mathrm{op}} \to \mathrm{GrAbGrp} \to \mathrm{Set}$$

for instance we can have $\mathrm{Vect}_{\mathbb{R}}$ which assigns to a topological space the set of all its real vector bundles, other relevant options here would be complex or oriented bundles. Note that this vector bundle functor sends a continuous map to pulling back by that map. Notice that our functors are contravariant, as in particular cohomology is contravariant (we could have pushed forward bundles).

A natural transformation is then a family of morphisms for any space B

$$\mathrm{Vect}(B) \to H^*(B) \in \mathrm{Set}$$

that satisfy the naturality conditions. In particular if we have a morphism of topological spaces $f : N \to M$, we need the following diagram to commute

$$\begin{array}{ccc} \mathrm{Vect}(M) & \xrightarrow{\mathrm{Vect}(f)} & \mathrm{Vect}(N) \\ \big\downarrow {\scriptstyle c(M)} & & \big\downarrow {\scriptstyle c(N)} \\ \mathrm{H}^*(M) & \xrightarrow{\mathrm{H}^*(f)} & \mathrm{H}^*(N) \end{array}$$

the commutativity of this diagram explicitly says that

$$f^* c(\xi) = c(f^* \xi)$$

this along with the fact that all bundle maps are pullbacks of some map on the base shows that axiom 2 is equivilent to naturality of this transformation.

Axioms $1, 3, 4$ do not follow, and require extra conditions on the type of (functor / natural transformation). If we think of Vect as a semigroup (set with associative operation) under Whitney sum and H^* as a semi-group under the cup product we get axiom 3 (that is we change the codomain of our functors to semigroup so the natural transformations must be morphisms of semi-groups). This also implies that $c_0^2 = 1$ and so we fix the convention that it is $+1$.

Finally note that natural transformations require the functors to have the same domain and codomain, however we really want cohomology to land in graded abelian groups or even rings, this suggests a need to put the same structures on the set of vector bundles.

2 Key Facts

Here are some key properties

1) The characteristic class of a trivial bundle is 1.

Consider a trivial bundle $B \times \mathbb{R}^n \to B$ then consider the map $\bar{f} : B \to *$. This induces a map $f : B \times \mathbb{R}^n \to * \times \mathbb{R}^n$ which is clearly a bundle map, to the trivial bundle over a point. Now apply natrality of the charactersitic classes we get that

$$c_i(B \times \mathbb{R}^n) = \bar{f}^* c_i(* \times \mathbb{R}^n) \in H^i(*)$$

and so much be zero for $i > 0$ and 1 in degree 0.

2) Isomorphic bundles (over the same base) have equal characteristic classes (more precisely their characteristic classes map to one another under the given isomorphism).

This is non-trivial as far as I can see. One first needs to prove that these classes exist and are unique. Given this however it might follow from the fact that isomorphisms induce isomorphisms in cohomology.

3) Stieffel-Whitney classes commute with (exterior) products:

$$w(\xi \times \xi') = w(\xi) \times w(\xi')$$

where the product on the right is the exterior product and on the left is the "Cartesian product", wich is a complex construction from pulling back pullbacks. Naively this can just be considered as a formal product of the cohomology classes, where the coefficients commute but the generators of the cohomology rings are different formal variables.

4) Chern classes and conjugation are well behaved:

$$c_k(\bar{\xi}) = (-1)^k c(\xi)$$

3 Key Examples

3.1 Sphere

The first easiest example is that of the sphere. We know that for the standard embedding

$$S^n \hookrightarrow R^{n+1}$$

the normal bundle is trivial, ϵ, thus applying the product rule we have that

$$1 = w(\epsilon) = w(\tau_{S^n} \oplus \nu) = w(\tau_{S^n})w(\nu) = w(\tau_{S^n}).$$

3.2 Complex Projective Space

This is a complex manifold with a complex tangent bundle. If we compute the Chern classes then we will know also the Euler class and the Steiffel-Whitney classes.

If we consider the model of $\mathbb{C}P^n$ as

$$\mathbb{C}P^n \quad := \quad \left\{ L \leq \mathbb{C}^n : \text{ rank 1 subspaces } \right\}$$

If we define the bundles

$$\begin{aligned} \gamma &:= \{(v, L) : v \in L\} \subseteq \mathbb{C}^{n+1} \times \mathbb{C}P^n \\ \gamma^\perp &:= \{(v, L) : w \in L^\perp \subseteq \mathbb{C}^n\} \end{aligned}$$

with the projection from the second variable. Note that γ is called *the cannonical bundle.* The tangent space of $\mathbb{C}P^n$ has the elegant expression in the form

$$\tau = \tau_{\mathbb{C}P^n} = \mathrm{Hom}(\tau, \tau^\perp)$$

where we consider Hom the bundle constructed fiberwise. Intuitively an element of this hom bundle is a map from the fiber at L which is just L to the normal to L, but L is one dimensional and so this is just an element of the normal space, i.e. a tangent vector.Actually would like to understand how is the space of lines topologised and how is that related to the cell structure.

We also know that $\mathrm{Hom}(\gamma, \gamma) \cong \epsilon$ that is a trivial bundle, because it has a no-where zero section given by the identity map at each point (thus it is parrallelizable) and is of dimension 1, each fiber is the point over which it is fiber, which are by definition rank 1 subspaces. Using that the trivial bundle has no effect on Chern number we get that

$$\begin{aligned} c(\tau) &= c(\tau \oplus \epsilon) \\ &= c(\tau \oplus \mathrm{Hom}(\gamma, \gamma)) \\ &= c(\mathrm{Hom}(\gamma, \gamma^\perp) \oplus \mathrm{Hom}(\gamma, \gamma)) \\ &= c(\mathrm{Hom}(\gamma, \gamma \oplus \gamma^\perp)) \end{aligned}$$

But $\gamma \oplus \gamma^\perp$ is a trivial rank $n+1$ bundle and so we get that

$$\begin{aligned} c(\tau) &= c(\mathrm{Hom}(\gamma, \mathbb{C}^{\oplus n+1}) \\ &= c(\mathrm{Hom}(\gamma, \mathbb{C})^{\oplus n+1}) \\ &= c(\bar{\gamma}^{\oplus n+1}) \\ &= c(\bar{\gamma})^{n+1} \end{aligned}$$

The $n+1$ fold cup product. Thus we get a binomial expansion in terms of the Chern class of this γ bundle. By our 4th axiom this is a generator of $H^2(\mathbb{C}P^n)$.

Now lets compute some example numbers of these spaces (the general case is in [MS16] but is just combinatorics from these basic ones), denote $\alpha = c_1(\bar{\gamma})$, then for example we know that

$$c(\mathbb{C}P^2) = (1+\alpha)^3 = 1 + 3\alpha + 3\alpha^2$$

noting that the final term α^3 from the binomial expansion is in $H^6(\mathbb{C}P^2) = 0$. α^2 is dual to the fundamental class of $\mathbb{C}P^n$, by convention it is dual on the nose, without a $-$ sign (they are both generators of the respective rank one modules and are therefore dual). Thus we can compute the

Chern numbers by considering partitions of 4 by even numbers (Chern classes only in even degrees), of which there are two $2+2$ and 4. These corespond to the Chern classes

$$c_1^2 = (3\alpha)^2, \quad c_2 = 3\alpha^2$$

which we can evaluate on $[\mathbb{C}P^2]$ and use its duality with α^2 to get immediately the associated Chern numbers

$$9, 3$$

respectively.

Now $\mathbb{C}P^1 \times \mathbb{C}P^1$ has Chern class

$$c(\mathbb{C}P^1 \times \mathbb{C}P^1) = c(\mathbb{C}P^1) \times (\mathbb{C}P^1) = (1+\alpha)^2(1+\beta)^2 = (1+2\alpha)(1+2\beta) = 1 + 2\alpha + 2\beta + 4\alpha\beta$$

it is of the same dimension, 4 so we have the same partitions and we can again compute two Chern numbers. By looking at the degree of cohomology the terms are in we recover which Chern charater the parts of the sum correspond to and we get

$$c_1^2 = (2\alpha + 2\beta)^2 = 4\alpha^2 + 8\alpha\beta + 4\beta^2 = 8\alpha\beta, \quad c_2 = 4\alpha\beta$$

the squares of the cohomology classes are zero for dimensional reasons. Now we again have by convention a duality between $\alpha\beta$ and the fundamental class thus the two respective numbers are

$$8, 4.$$

Lets go up a dimension and compute Pontryagin classes and numbers. If we didnt go up a couple of dimensions (4) there would only be one number, not very interesting. Lets do similar examples of the 8 manifolds $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$. A key fact is that for a complex manifold with a complex tangent space we have that

$$\tau \otimes \mathbb{C} \cong \tau \oplus \bar{\tau}$$

thus using the product property and denoting the tangent bundle of $\mathbb{C}P^4$ as τ we get that

$$\tilde{p}(\tau) = c(\tau)c(\bar{\tau}) = (1+\alpha)^5(1-\alpha)^5 = (1-\alpha^2)^5 = 1 - 5\alpha^2 + 10\alpha^4 - 10\alpha^6 + 5\alpha^8 - \alpha^{10} = 1 - 5\alpha^2 + 10\alpha^4$$

where the higher terms are zero for dimension reasons. In otherwords

$$p(\tau) = 1 + 5\alpha^2 + 10\alpha^4$$

We need partitions of 8 that are in degrees 4 and 8 of which there are again (essentially the same) thus we have the two classes of interest

$$p_1^2 = 25\alpha^4, \quad p_2 = 10\alpha^4$$

which gives the numbers $25, 10$.

For $\mathbb{C}P^2 \times \mathbb{C}P^2$ we have the Pontryagin class of the tangent space given by

$$\tilde{p}(\tau) = \tilde{p}(\tau_{\mathbb{C}P^2}) \times \tilde{p}(\tau_{\mathbb{C}P^2}) = (1+\alpha)^3(1-\alpha)^3(1+\beta)^3(1-\beta)^3 = (1-3\alpha^2)(1-3\beta^2) = 1 - 3(\alpha^2+\beta^2) + 9\alpha^2\beta^2$$

Again in other words we have that

$$p(\tau) = 1 + 3(\alpha^2 + \beta^2) + 9\alpha^2\beta^2$$

The two classes are then

$$p_1^2 = 9(\alpha^2+\beta^2)^2 = 9(\alpha^4 + 2\alpha^2\beta^2 + \beta^4) = 18\alpha^2\beta^2, \quad p_2 = 9\alpha^2\beta^2$$

thus the numbers are $18, 9$ respectively.

	$\mathbb{C}P^2$	$\mathbb{C}P^1 \times \mathbb{C}P^1$
c_1^2	9	8
c_2	3	4

	$\mathbb{C}P^4$	$\mathbb{C}P^2 \times \mathbb{C}P^2$
p_1^2	25	9
p_2	10	1

4 The Cobordism Ring

If we denote Ω_n the collection of smooth compact oriented n-dimensional manifolds identified up to cobordism then it becomes an additive group under disjoint union.

Theorem. *$(\Omega_n, +)$ is a finite group for $n \not\equiv 4$ and is a finitely generated rank $\mathcal{P}(n/4)$ group when $n \equiv 4$.*

SKETCH PROOF? This is highly non-trivial it is the final chapter of the book. Note that the torsion part is not classified by this theorem when $n \equiv 4$. ⚠️ We are being circular in stating these theorems first, however conceptually it is clearer.

This additive group is also a *graded ring* under cartesian product. We denote this graded ring Ω_*.

Lemma. *If a $4k$ manifold is a boundary then all of its Pontryagin numbers are zero.*

SKETCH PROOF? In particular we have for any partition I a group homomorphism

$$\Omega_{4k} \to \mathbb{Z}$$

$$M \mapsto p_I(M)$$

or equivilently we have a homomorphism

$$\Omega_{4k} \to \mathbb{Z}^{\mathcal{P}(k)}$$

$$M \mapsto (p_I(M))_I$$

That is a $\mathbb{Z}$ -module homomorphism from a rank $\mathcal{P}(k)$ module to a rank $\mathcal{P}(k)$ module, which can be represented as a matrix. Because we want to disregard the torsion factors we may as well start talking about $\Omega_* \otimes \mathbb{Q}$ and so this gives us a linear map between vector spaces of the same dimension. In particular it can be represented by a matrix.

Given some manifolds $K^1, ..., K^n$ (of dimension n or $4n$) then *under some mild hypothesis* the Sort out the hypothesis.$\mathcal{P}(n) \times \mathcal{P}(n)$ matrix (the number of partitions of n)

$$\left[c_{i_1} \cdots c_{i_r}[K^{j_1} \times K^{j_s}]\right]$$

where $i_1, ..., i_r$ and $j_1, ..., j_s$ range over all partitions of n is non-singular.SKETCH PROOF? In particular the theorem is satisfied by the complex projective spaces. Indeed we can see that the columns of our two matrices in the computations above are in fact linearly independent (and that they are the relevant matrices).

This matrix is exactly the $\mathbb{Q}$ module homomorphism $\Omega_{4k} \otimes \mathbb{Q} \to \mathbb{Q}^{\mathcal{P}}(k)$ given by picking a some elements on the left, namely all the different products of the sequence of $K^1, ..., K^n$, and this statement says that it is a surjection. But a surjection between two vector spaces of the same dimension is an isomorphism. Thus it shows that these products actually form a basis as a $\mathbb{Q}$ vector space for the cobordism group in the relevant dimension. In particular products of (even dimensional) complex projective spaces form such a basis.

5 Signature

Let M be a compact, oriented $4k$ manifold with fundamental class μ. Then the cohomology over a field, say $\mathbb{Q}$, in the middle dimension has a bilinear form by linearly extending

$$H^{2k}(M;\mathbb{Q}) \otimes H^{2k}(M;\mathbb{Q}) \to \mathbb{Q}$$

$$(a,b) \mapsto (a \smile b)(\mu)$$

the evalutation of their cup on the fundamental class as before. It is well known that this is a non-singular form ([Hat02, Prop 3.38]. The dimension being a multiple of 4 ensures that this form is symmetric. Together these imply that this form is represented by an invertible diagonalisable matrix. The signature of the manifold is then given by the signature of the quadratic form, or explicitly

$$\sigma(M) \quad := \quad \# \text{ of positive diag entries} - \# \text{ of negative diag entries}$$

or equivilently the number of positive minus the number of negative eigenvalues of the form.

This has the key properties

1) This is an invariant of the space. This is clear because the cohomology ring and fundamental class are invariants of the space.

2) (Sums)

$$\sigma(M \sqcup M') = \sigma(M) + \sigma(M')$$

We have field coefficients as we can see the homology of a disjoint union is the direct sum of the homologies (Mayer-Vietoris), which therefore also applies to cohomology. Therefore the matrix we are diagonalising is the direct sum of the two matricies and so the signature will add.

3) (Products)

$$\sigma(M \times M') = \sigma(M)\sigma(M')$$

The Kunneth isomorphism gives us the following commuting diagram

$$\begin{array}{ccc} H^*(M) \otimes H^*(M) \otimes H^*(M') \otimes H^*(M') & \xrightarrow{(-\smile-)[M]\cdot(-\smile-)[M']} & \mathbb{Q} \\ \downarrow & & \| \\ H^*(M \times M') \otimes H^*(M \times M') & \xrightarrow[(-\smile-)[M\times M']]{} & \mathbb{Q} \end{array}$$

where we have used the maps

$$H_n(M) \to \mathrm{Hom}(H_n(M), \mathbb{Q}) \to H^n(M)$$

$$\alpha \mapsto \big(\alpha \mapsto 1\big) \mapsto [\big(\alpha \mapsto 1\big)]$$

which are all isomorphis and then moved the fundamental class around this diagram

$$\begin{array}{ccc} H^n(M \times M') & \underset{\sim}{\longleftrightarrow} & H^n(M) \otimes H^n(M') \\ \updownarrow \sim & & \sim \updownarrow \\ H^n(M \times M') & \overset{\sim}{\dashleftarrow\dashrightarrow} & H_n(M) \otimes H_n(M') \end{array}$$

Im still not totally clear on how to show that this diagram commutes formally

4) If M is oriented, or if we choose the opposite fundamental class then we have the relation

$$\sigma(M) = -\sigma(-M)$$

this is clear.

5) If M is a boundary then $\sigma(M) = 0$.

We will use Poincare duality as well as a linear algebra lemma:

Lemma. *If $\lambda : V \times V \to \mathbb{Q}$ is a non-singular symmetric bilinear form and the rank of V is 2ℓ, then if there exists an ℓ dimensional subspace $L \leq V$ such that $\lambda|_{L\times L} = 0$ then $\sigma\lambda = 0$.*

Such a subspace is called a *Lagrangian* of the bilinear form. Clearly the hypothesis on the form are fullfilled for our pairing so what we need to show is that if $M^{4k} = \partial X$ then there is a Lagrangian of $H^{2k}(M)$. Consider the LES in cohomology

$$\cdots \to H^{2k}(X, M) \to H^{2k}(X) \xrightarrow{i^*} H^{2k}(M) \to \cdots$$

where $i : M \to X$ the inclusion. We claim that $\mathrm{Im}(i^*)$ is a Lagrangian for the relevant pairing that we now denote λ. First we need the form to vanish. Consider the commuting diagram

$$\begin{array}{ccc} H^{2k}(X) \otimes H^{2k}(X) & \xrightarrow{\smile} & H^{4k}(X) \\ & & \downarrow i^* \\ \downarrow i^* \otimes i^* & & H^{4k}(M) \\ & & \downarrow ev_{[M]} \\ H^{2k}(M) \otimes H^{2k}(M) & \xrightarrow[\lambda]{} & \mathbb{Q} \end{array}$$

this commutes because of natrality

$$(a \smile b) \circ i_* = ai_* \smile bi_*.$$

The right hand vertical is zero however since I feel like it should be because we are evaluating an X class on the fundamental class of M and X is 4k+1 dimensional. Clarify...

Now we show that this image has the correct dimension, namely $\frac{1}{2}H^{2k}(M)$. We compare the LES in homology with that in cohomology using Poincare duality for manifolds with boundary which gives us iso's in every place

$$
\begin{array}{ccccccccc}
H^{2k}(W,M) & \longrightarrow & H^{2k}(W) & \xrightarrow{i^*} & H^{2k}(M) & \xrightarrow{j^*} & H^{2k+1}(W,M) & \longrightarrow & H^{2k+1}(W) \\
 & & \wr\| & & \wr\| & & \wr\| & & \\
H_{2k}(W) & \longrightarrow & H_{2k+1}(W,M) & \longrightarrow & H_{2k}(M) & \xrightarrow{i_*} & H_{2k}(W) & \longrightarrow & H_{2k-1}(W,M)
\end{array}
$$

Because we are in the category of vector spaces we know that

$$H^{2k}(M) \cong \text{Im}(j^*) \oplus \ker(j^*) \cong \text{Im}(j^*) \oplus \text{Im}(i^*)$$

by rank nullity and exactness. Thus it is sufficient to show that the $\text{Im}(i^*) = \text{Im}(j^*)$ as then the dimension of H^{2k} would be $2\text{Im}(i^*)$ as required. Using our Poincare duality diagram (particularly its commutativity) however we get that

$$\text{Im}(j^*) \cong \text{Im}(i_*) \cong \text{Im}(i_*)^* \cong \text{Im}(i^*)$$

recalling that vector spaces are isomorphic to their duals and applying universal coefficients for the final iso.

In particular the signature is a cobordism invariant. Thus it gives a ring homomorphism

$$\sigma : \Omega_* \to \mathbb{Z}$$

Thus we have shown that

Theorem (Signature Theorem, [MS16], Thm 19.4)**.** *The signature of a manifold is a linear combination of the Pontryagin numbers.*

This is clear becuase we essentially showed that the Pontryagin numbers give an isomorphism from

$$\Omega_{4k} \otimes \mathbb{Q} \to \mathbb{Q}^{\mathcal{P}(k)}$$

thus any group homomorphism from $\Omega_{4k} \to \mathbb{Q}$ (namely signature) will factor through this isomorphism.

Remark: The full power of this theorem is that it gives an exact formula in terms of the tanh of the Pontryagin numbers. We will see this later.

5.1 Example

Lets compute the signature directly for low dimensional complex projective spaces.

Now we can use our Pontryagin number computations from earlier to find *which* linear combination of them gives us the signature.

Now for instance if we have another 8 manifold and we know two out of three of signature, p_1 or p_2 then we can sometimes sub it into these linear relations and find the third. For example $\mathbb{H}P^2$ the quarturnionic projective space we are told that $\sigma = 1$ and $p_1 = \pm 2\alpha$ for a generator α in H^4. Then using that $p_2 = d\alpha^2$ we can solve for $d = 7$ to get the second Pontryagin class. Note that we used a lot of facts. A priori we didnt have two out of the three, we had to use the duality of α^2 and the fundamental class as well to get the pontryagin number of p_1^2 and then leverage this to get the number for p_2 and then the coefficient for p_2 in terms of p_1.

6 Genera

The study of genera is going to unify topology and complex analysis and through complex analysis we also get all the connections with number theory.

6.1 Multiplicative Sequences

Throughout we will use R as a fixed commutative unital ring. A^* is a (strictly / classically) commutative graded R algebra. One should think of $\mathbb{Q}$ and H^*. To A^* we associate the commutative ring of sequences (power series) $(a_0, ..., a_n, ...) \in A^\Pi$ with power series multiplication. We call a sequence of homogeneous polynomials with coefficients in R

$$K_1(x_1), K_2(x_1, x_2), \cdots$$

where K_n is homogeneous (terms have the same degree) of degree n, *multiplicative* if for all A^*, graded R algebras and all $a, b \in A^\Pi, a_0 = b_0 = 1$ denoting

$$K(a) \quad := \quad 1 + K_1(a_1) + K_2\big(a_1, a_2\big) + \cdots$$

we have that

$$K(ab) = K(a)K(b).$$

Remark: Homogeneous here means *weighted* homogeneous, that is the terms in say $K_n(x_1, ..., x_n)$ should all have the same degree, where degree is calculated by

$$\text{Deg} = \sum_i i \cdot \text{exponent}(x_i)$$

eg. the degree of x_3 is 3, which is the same as the degree of $x_1 x_2$.

Hirzebruch classified these sequences by showing that they are in one-one correspondence with power series.

Theorem. *If f is a formal power series with coefficients in R and constant term 1 then there exists a unique multiplicative sequence $\{K_n\}$ with R coefficients satisfying*

$$f(t) = K(1 + t)$$

Note that we are considering $1 + t$ as an element of the graded R algebra $R[t]$ or as an element of $R[t]^\Pi$. This multiplicative sequence we call the multiplicative sequence *belonging* to the power series. It has two properties

- If $f(t) = 1 + \sum_i \lambda_i t^i$ then K_n is the associated multiplicative sequence iff the coefficient of x_1^n in K_n is λ_n.
- If K_n comes from f then for any $a^1 \in A^1 \subseteq A^*$ degree one element of any graded R algebra we have the identity

$$K(1 + a_1) = f(a_1)$$

Example. *//*

Remark: First of all it is clear that $K(1 + t)$ is a power series, the uniqueness claim in the theorem shows that if two multiplicative sequences agree as power series

$$K(1 + t) = K'(1 + t)$$

then the multiplicative sequences are equal. This shows the injection in the other direction.

6.2 Proof Sketch

the correspondence is given by

$$1 + \sum_i \lambda_i t^i \mapsto K_n....notclear$$

the way they define it is not so clear, why are they defining it on a basis of the symetric functions? Well the wikipedia on genus of multiplicative sequence says something about how to do this, regardless it seems that the definition is not so straight forward; it is trying to basically just take all the combinations of suitable degree of the coefficients of the power series.

This is key to understanding how Hirzebruch would come up with the multiplicative sequence for the signature.

6.3 Genus

Now to a smooth compact oriented $4n$-dimensional manifold M and a multiplicative sequence $\{K_n\}$ over the rationals we call the K genus of M

$$K_n[M] \quad := \quad K_n(p_1, ..., p_n)[M]$$

where p_i are the Pontryagin classes of the tangent bundle of M. Notice that K_n is a polynomial and that each of the summands has the same degree, if we evaluated any one of the summands of K_n then we would get simply a Pontryagin number. Here however we are taking $\mathbb{Q}$ linear combinations of such things.

Lemma. *Given a multiplicative sequence K_n over $\mathbb{Q}$ there is an algebra homomorphism*

$$\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$$

$$M^{4n} \mapsto K_n[M]$$

Proof. Because it is a linear combination of Pontryagin numbers linearity is clear (module homomorphism). Thus all we need to check is that it respects products. When proving the Whitney product formula for Stieffel-Whitney classes the first step is to compute the Stieffel-Whitney class for the cartesian product and then find the formula for the Whitney sum. Something *similar* but less straight forward is done for Chern classes, thus we still need a more thorough analysis of the Pontryagin (Chern) classes of a Cartesian product.

The claim is that just like for the Whitney sum we have that

$$p(M \times M') \equiv p(M) \times p(M') \pmod 2$$

that is modulo elements of order 2 in the cohomology ring. I dont think they supply a proof of this maybe its an easy argument similar to the Whitney sum one?. Then mod 2 we have

$$K_{n+n'}(p(M \times M')) \equiv K_n(p(M)) \times K_{n'}(p(M'))$$

Thus rationally we have the classes are equal (in $\mathbb{Q}$ coefficient cohomology) and so we conclude that

$$K_{n+n'}(p(M \times M'))[M \times M'] = K_n(p(M))[M] \cdot K_{n'}(p(M'))[M'] \in \mathbb{Q} \tag{1}$$

Remark: Milnor-Stasheff says something a bit different; they say that

$$K(p \times p')[M \times M'] = (-1)^{mm'} K(p)[M]K(p')[M']$$

both Diarmuid and I were confused by the sign conventions being used here. Regardless the power is even and so the proof still goes through.

Remark: Notice we used multiplicative property of K in equation 1.

Remark: By convention if the manifold does not have dimension $4k$ its signature and K genus are defined to be 0.

6.4 Back to Signature

We have already seen that the signature is a rational linear combination of the Pontryagin numbers, the claim is now that this linear combination arrises from a multiplicative sequence.

Theorem (Signature Theorem (Again))**.** *The signature is the L genus for the multiplicative sequence corresponding to the Taylor expansion of*

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2^{2k} B_k t^k}{2k!}$$

Note that B_k are the Bernoilli numbers with some convention choices. Namely Milnor-Stasheff have defined them to be the coefficients appearing in this expansion of $x/\tanh x$. Note also that this is a property of the signature, thus *any* manifolds signature will be *the same* linear combination of its Pontryagin numbers, it is however the Pontryagin numbers that will change.

Unpacking this a bit more what do we really get. We get that the signature as a map $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$ factors through the pieces $\Omega_{4n} \otimes \mathbb{Q} \to \mathbb{Q}$ and on each of these peices the signature is a homogeneous polynomial in the Pontryagin classes (evaluated on the fundamental class) or that it is a linear combination of Pontryagin numbers, where the linear combination is independent of the manifold. In particular the linear combination is fixed by some power series. Calculating the L polynomials is in principle possible to do purely combinatorially if you know the coefficients of the power series (which we do), in particular there are explicit and recursive formulas for the Bernoulli numbers.

Proof. We are trying to show two algebra homomorphisms agree and so it suffices to check they agree on a set of generators. In particular we have shown that $\mathbb{C}P^{2j}$ generate the cobordism ring rationally.

The signature of all of these spaces is 1 because $H^{2j}(\mathbb{C}P^{2j})$ is generated by α and $\alpha^2[\mathbb{C}P^{2j}] = 1$. Thus as a 1×1 matrix the bilinear form of cupping is just 1. Note that there is a sign convention for the fundamental class, that also propegates to the L polynomials.

Now we just need to show that

$$L(\mathbb{C}P^{2j}) = 1.$$

First recall the general formula (derived easily from that we computed for Chern classes) for Pontryagin classes of the tangent space to $\mathbb{C}P^{2j}$ is given by

$$p = (1 + \alpha^2)^{2j+1}$$

for the same α as above. By definition we have that

$$L(1 + \alpha^2) = \frac{\sqrt{\alpha^2}}{\tanh\sqrt{\alpha^2}}$$

where the RHS is notation for the formal power series given by its Taylor series. By considering $1/\tanh = \coth$ taking the standard power series for coth and multiplying by t we get that

$$\frac{\sqrt{t^2}}{\tanh\sqrt{t^2}} = \frac{t}{\tanh t}$$

so we may write

$$L(1+\alpha^2) = \frac{\alpha}{\tanh\alpha}$$

By multiplicativity we get that

$$\begin{aligned}
L(p) &= L((1+\alpha^2)^{2j+1}) \\
&= L(1+\alpha^2)^{2j+1} \\
&= \Big(\frac{\alpha}{\tanh\alpha}\Big)^{2j+1} \\
&= \left[1+\sum_{k=1}^{\infty}\frac{\big(-1\big)^{k-1}2^{2k}B_k\alpha^{2k}}{2k!}\right]^{2j+1} \\
&= 1 + L_1(p_1) + L_2(p_1,p_2) + \cdots
\end{aligned}$$

$\mathbb{C}P^{2j}$ is real dimension $4j$ and so its relevant L genus will be L_j. L_j is a homogeneous polynomial of degree j in the Pontryagin classes, thus lands in the cohomology of degree $4j$. α above is a generator for H^2 and hence $\alpha^{2j} \in H^{4j}$ is a generator. Comparing coefficients and evaluating on a fundamental class shows that $L_j(p)[\mathbb{C}P^{2j}]$ is just the coefficient of α^{2j} in the power series

$$(\alpha/\tanh\alpha)^{2j+1}$$

What remains is to show that the coefficient of z^{2k} in $(z/\tanh z)^{2k+1}$ is one. We consider z to be a complex variable. Then away from 0, where $\tanh(0) = 0$ we have the ratio of holomorphic functions and thus the function is holomorphic. tanh has a zero of order 1 at 0 and hence $1/\tanh$ has a pole of order 1 at 0. Thus $z/\tanh z$ is holomorphic at 0, that is to say entire. Its Taylor series at 0 is the one we have been considering, or otherwise it is clear that it is holomorphic in an anulus around 0 and hence has a Laurent series. Now by Laurents theorem or the definition of Laurent series we have that for a complex function (around zero)

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$$

where a_n is given by the contour integral around 0 on a closed curve

$$a_n = \frac{1}{2\pi i}\oint \frac{f(z)}{z^{n+1}}$$

thus for us the relevant coefficient is given by

$$a_{2k} = \frac{1}{2\pi i}\oint \frac{z^{2k+1}}{z^{2k+1}(\tanh z)^{2k+1}} = \frac{1}{2\pi i}\oint \frac{1}{(\tanh z)^{2k+1}}$$

Now its a complex analysis exercise. First we substitute $u = \tanh z$, so $z = \operatorname{arctanh}(u)$ and

$$\frac{dz}{du} = \frac{1}{1-u^2}$$

which by a geometric series is equal to

$$\frac{1}{1-u^2} = \sum_{i\geq 0} u^{2i}$$

Note that we can take our curve in a neighbourhood of 0 , since $\tanh(0) = 0$ and so this formula applies. Thus the contour integral becomes

$$\begin{aligned}
\frac{1}{2\pi i}\oint \frac{1}{(\tanh z)^{2k+1}} &= \frac{1}{2\pi i}\oint \frac{\sum_{i\geq 0} u^{2i}}{u^{2k+1}} \\
&= \frac{1}{2\pi i}\sum_{i\geq 0}\oint u^{2i-2k-1} \\
&= \frac{1}{2\pi i}\sum_{i\geq 0} 2\pi i \mathrm{Res}_0 u^{2i-2k-1} \\
&= \sum_{i\geq 0}\begin{cases} 1, & 2i-2k-1=-1 \\ 0, & \text{else} \end{cases} \\
&= 1
\end{aligned} \tag{2}$$

where we assume that there is no problem with interchanging the sum but will not check seriously. Recall that the residue of a holomorphic function is the coefficient of the U^{-1} term of its Laurent expansion and that a polynomial is its own Laurent expansion.

Thus the L-genus given by $\sqrt{t}/\tanh\sqrt{t}$ is constantly one on projective spaces and thus agrees with the signature.

Remark: More enlightening would be the following exercise; how would we construct a power series such that its multiplicative sequence always gives 1.

References

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge ; New York, 2002.

[Hir66] Friedrich Hirzebruch. *Topological Methods in Algebraic Geometry.* Springer, Berlin, Heidelberg, 1966.

[MS16] John Willard Milnor and James D. Stasheff. *Characteristic Classes: AM-76.* Number 76 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2016.